

ANALYSIS OF HEMT MULTILAYERED STRUCTURES USING A 2D FINITE VOLUME MODEL

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ABSTRACT

This paper uses a 2D finite volume numerical model to predict the steady state and transient temperature distributions in a High-Electron-Mobility Transistor (HEMT). The numerical predictions are confirmed through comparison with analytical solutions of the one and two dimensional steady and transient heat equations. We analyze the thermal performance of several HEMT geometries with varying substrate materials. Devices with wider gates lying directly on highly conductive substrates (e.g., diamond) have significantly less thermal resistance, by as much as 90 percent. Finally, we investigate the temperature response to a frequency modulated heating event. This result indirectly applies to thermal measurements using 3ω electrical thermometry.

KEY WORDS: High Electron Mobility Transistors (HEMT), Thermal Resistance, Multilayered thin film structures, GaN devices.

NOMENCLATURE

The schematic of a typical HEMT device analyzed in this paper is shown in figure 1. It consists of a gate, which is the main source of heat and n layers of various films, with the last layer being the substrate. The nomenclature used in this paper is as follows:

A	diffusion cross section (m^2)
b	stack depth into the page, μm
C	volumetric heat capacity of medium ($\text{J K}^{-1} \text{m}^{-3}$)
c	sound velocity (m/s)
d	total device thickness (10^{-6}m)
k	thermal conductivity ($\text{W m}^{-1} \text{K}^{-1}$)
L	structure half width (10^{-6}m)
q	total input heat flux (W)
R	thermal resistance (K W^{-1})
T	temperature (K)
t	time (s)
u	$T - T_0$ (K)
w	gate length (10^{-9}m)
x	position along x coordinate axis (10^{-9}m)

y position along y coordinate axis (10^{-9}m)

Greek Symbols

α thermal diffusivity, m^2/s

Subscripts

D	average over all phonon polarizations
i	material i
L	longitudinal
store	refers to whole stack
T	transverse
t	first derivative in time
xx	second derivative in x-direction
yy	second derivative in y-direction

INTRODUCTION

High Electron Mobility (HEMT) devices are attractive because of their geometrical and performance advantages over Metal Semiconductor Field Effect Transistors (MESFETs) [1]. The reliability and performance of GaN devices depend critically on the operating temperature, which drives the design of the cooling system and the input power limitations of the device [2].

In this paper, our goal is to compare the thermal performance of various HEMT structures with different substrates and HEMT-substrate thermal boundary resistances (TBRs). The result of this investigation helps us understand the impact of various substrates on the operating temperatures of HEMT devices. Further, such models can indirectly extract the thermal properties of a HEMT device from surface temperature measurements.

The temperature distribution is well predicted by the heat equation in three dimensions. However, only a few cases can be solved analytically in closed form [2]. The presence of multiple layers, each with different thermal properties and thermal boundary resistances, complicates any analytical solution and can lead to very complicated coefficient expressions, e.g., for solutions obtained using separation of variables.

Feldmen [3] proposed an algorithm to solve the one dimensional heat equation in a multilayered stack subject to

AC heating. A problem with this method is that the effect of gate length on stack thermal resistance cannot be directly examined since this method assumes spacially uniform heat flux along the heating interface. Further, the Feldman method is intended for AC heating events and application to other heating profiles requires some manipulation.

Darwish et al [2] proposed a resistance model based on solving the steady state heat equation in cylindrical, prolate spheroid, and elliptic cylindrical coordinate systems. Although this method returns a very good approximation for maximum operating temperature, it is only valid for predicting the steady-state behavior of a two-layer device (GaN and a substrate). In order to fully investigate the effects of gate length and substrate material on devices with arbitrary material structures, we need to develop a numerical solution.

A DISCUSSION ON THERMAL BOUNDARY RESISTANCES

In order to analyze the thermal behavior of multilayered stacks, we must determine the thermal boundary resistances between various layers of the stack. One way to gain such insight is to use the diffuse mismatch model suggested by De Bellis et al [4]. According to this model, in order to compute the thermal boundary resistance between material 1 and 2, we use:

$$R = \left(\frac{a_{1 \rightarrow 2}}{12} c_{1D}^3 \sum_j c_{1j}^{-2} \right)^{-1} C_1(T) \quad (1)$$

Where phonon transmissivity is:

$$\alpha_{1 \rightarrow 2} = \frac{\sum_j c_{2j}^{-2}}{\sum_j c_{1j}^{-2} + \sum_j c_{2j}^{-2}} \quad (2)$$

c_{ij} is the phonon velocity of material i in the j^{th} direction, c_{1D} is the average phonon velocity in material 1:

$$c_{1D} = \left(\sum_j \frac{c_j^{-3}}{3} \right)^{-\frac{1}{3}} \quad (3)$$

And C_1 is the volumetric heat capacity of material 1. Table 2 catalogs the phonon velocities and resulting analytical TBR solutions.

Table2: Phonon velocities in various materials.

Material	Phonon Velocities (m/s)		R(m ² K/GW)
GaN[5]	c_{T1}	4427	R(GaN->AlGaN)=0.5
	c_{T2}	4427	
	c_L	7781	
AlGaN [6]	c_{T1}	4000	R(AlGaN->AlN)=1.34
	c_{T2}	4000	
	c_L	9000	

AlN[5]	c_{T1}	6419	R(AlN->Diamond)=0.5
	c_{T2}	10986	
	c_L	10986	
Diamond [7]	c_{T1}	12830	R(GaN->Diamond)=3.7
	c_{T2}	16600	
	c_L	18330	
SiC[8]	c_{T1}	7250	R(GaN->SiC)=1.31
	c_{T2}	7250	
	c_L	13260	
Si [9]	c_{T1}	5332	R(GaN->Si)=0.8
	c_{T2}	5332	
	c_L	8970	

The values obtained using the diffuse mismatch model assume ideal conditions at the interface. Since a perfect interface (i.e. an interface with no grain boundaries, no stresses, no impurities and no interdiffusion) is very difficult to create, the actual thermal boundary resistances are going to be higher than the above values. Kuball et Al [10] obtained resistance values for the GaN-SiC interface ranging from 10 to 60 m²K/GW at elevated temperatures. In a paper submitted to this conference [11], a resistance value of 20 m²K/GW is reported for the AlN-diamond interface. Hence, we have reasonable grounds to assume higher resistance values than the ones calculated here.

Thus, in our simulations, we assume that the boundary resistance for GaN-AlGaN and the AlGaN-AlN interface to be 5 and 10 m² K/GW respectively. We then vary the bottom thermal boundary resistance of each stack within a reasonable range to investigate the effect of various substrates on thermal performance of the device. The lower limit of this range is the value obtained in table 2. The higher limit for the range is the highest value in the literature. If such an upper limit is not available, we set a limit similar to those found in the literature for comparable interfaces.

MODEL SPECIFICATIONS

This simulation is written in the MATLAB environment. It solves the two-dimensional transient heat equation (equation 4) in Cartesian coordinates using the finite volume method, and handles the steady state problems as a special case.

$$\alpha(u_{xx} + u_{yy}) = u_t \quad (4)$$

It should be noted that the two-dimensional heat transfer assumption is only valid for cases in which the depth of the stack into the page (b) is much larger than the gate spacing (2L) and the stack depth (d) (Fig. 1). The typical geometry handled by our model is shown below. All surfaces except for the area below the heater and the bottom surface are assumed to be adiabatic. The model can handle both convection and constant temperature boundary conditions at the bottom surface.

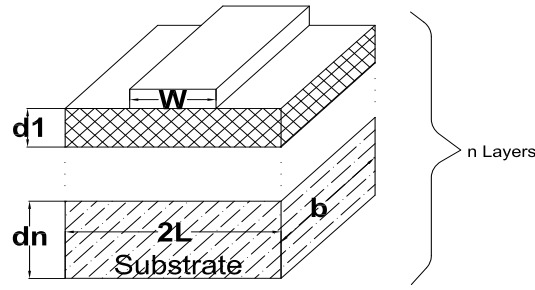


Figure 1: 3D schematic of a typical HEMT device.

It should be noted that this model resembles a multifinger transistor. Thus, adiabatic boundary conditions on the sides are imposed due to symmetry. In the case of a single figure device, we achieve better thermal performance since the adiabatic boundary conditions would no longer be necessary.

This model utilizes a central difference FD scheme to approximate spatial second derivatives. Moreover, it uses the first order implicit Euler scheme to calculate the temporal derivative, thus ensuring the stability of the solution. Table 1 lists the material properties used in constructing this model.

VALIDATION

First, we need to validate the code using limiting cases and compare the results of the simulation to the known analytical solutions for those cases.

Table1: Material properties used in this paper.

Material	Thermal Conductivity (k)	Thermal Diffusivity (α)
GaN	80 W/mk	3.08e-5
GaAlN	16 W/mk	
AlN	30W/mk	
Diamond	1800 W/mK	1e-3
Si	150 W/mK	
SiC	250 W/mK	

a) One-Dimensional Steady State Validation:

Figure 2 demonstrates the configuration of our 1-D HEMT model validation structure. It should be noted that in order to reach the 1D limiting case, we require uniform heating along the entirety of the top surface. That is, the length of the gate (w) must be equal to the device width (2L).

For the power input, we refer to the case study by Darwish et al [2]. There, a total power input of 20V x 670 mA was assumed for a device with 8 gates. Hence, we take an eighth of that value (1.675W) to be the power input for our single gate. Figure 3 shows the simulation result for this limiting case.

The numerical 1-D limiting case agrees well with the 1-D thermal resistor model. The total predicted temperature rise

(102.175K) closely matches the analytical resistance model temperature rise (102.091K).

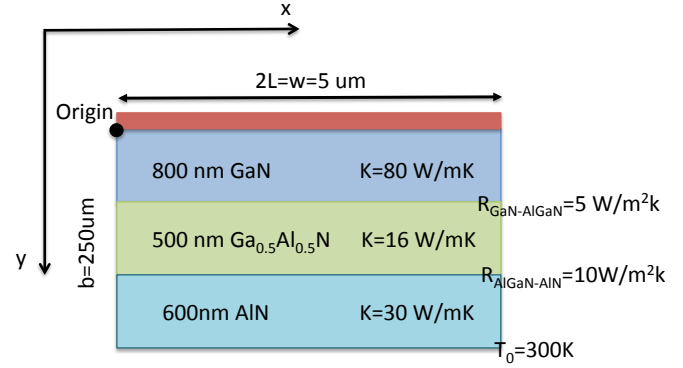


Figure 2: A typical multilayered HEMT structure. The GaAlN and AlN films act as transition layers to promote single-crystal GaN growth. The substrate, which sits below the AlN layer, has been replaced with a constant temperature boundary condition for initial validation purposes. This figure also shows the x and y coordinate axes along with the origin of the coordinate system.

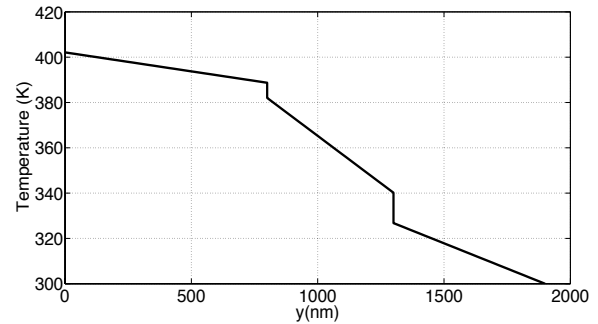


Figure 3: Temperature (K) vs. y (nm) plot for the stack shown in figure 2.

b) One Dimensional Transient Validation:

Having validated the 1D steady-state case, we now seek to confirm the accuracy of the transient solution generated by our numerical model. For this purpose we compare the simulation results to the analytical solution of the one dimensional transient heat equation. The appendix details the derivation of the analytical solution. The device configuration is the same as for the steady-state one-dimensional case, where the length of the gate is equal to the device width. To minimize the calculation time for the validation, we simplified the device geometry (Fig. 4). Figure 5 shows the results of this simulation, with the analytical and numerical values being practically indistinguishable.

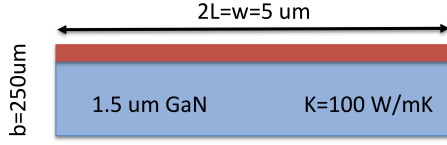


Figure 4: Geometry used for one-dimensional transient and two dimensional steady state and transient validation.

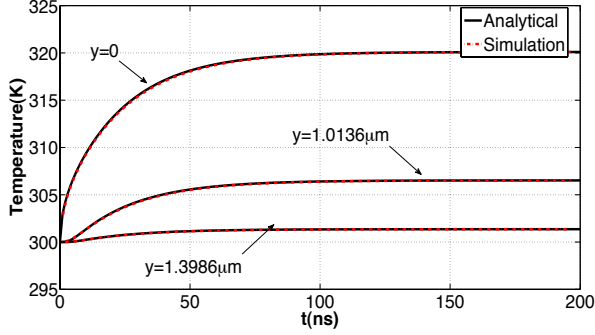


Figure 5: Temperature (K) vs. t (ns) for various depths. This plot illustrates the analytical solution and the simulation results of a 1D transient problem.

c) Two-Dimensional Steady State Validation:

As in the previous cases, we compare the analytical solution of a sample heat equation to our simulation results. We consider the geometry of figure 4 with a triangular heating profile along the top surface. The heat equation and boundary conditions are shown in the appendix. The following figures demonstrate that the simulation and analytical solutions are in agreement.

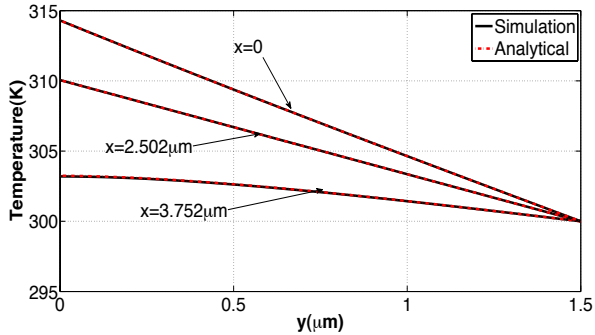


Figure 6a: Temperature (K) vs. y (μm) for various constant x values. This plot illustrates the analytical solution and the simulation results of a 2D steady state problem with triangular input heat distribution along the top surface.

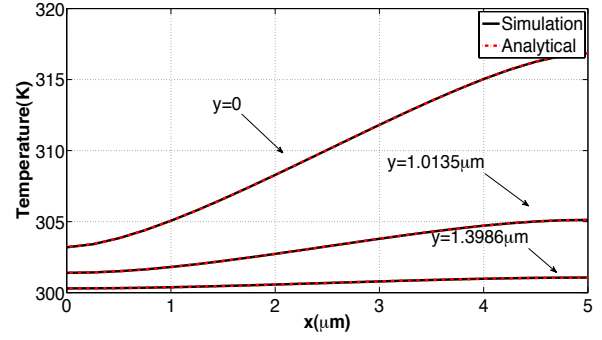


Figure 6b: Temperature (K) vs. x (μm) for various constant y values. This plot illustrates the analytical solution and the simulation results of a 2D steady state problem with triangular input heat distribution along the top surface.

d) Two-Dimensional Transient Validation:

We now generalize our simulation to the solution of the transient two-dimensional heat equation, considering the geometry of figure 4 and a triangular input heat distribution. Figure 7 shows excellent agreement between the analytical and numerical solutions

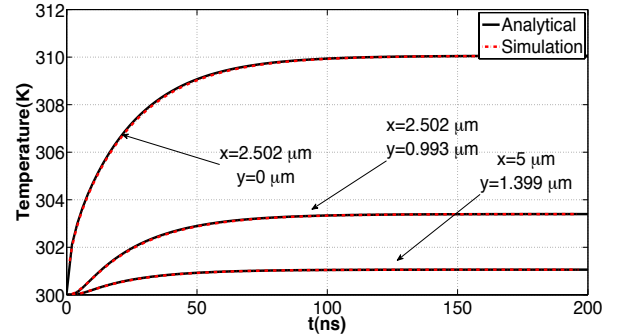


Figure 7: Temperature (K) vs. t (ns) for various points. This plot illustrates the analytical solution and the simulation results of a 2D transient problem with triangular input heat distribution along the top surface.

RESULTS

a) Thermal resistance across stack:

To analyze the total thermal resistance across a typical HEMT structure (fig. 8), we apply heating power to the gate and measure the average temperature drop below the gate at steady state. Figure 9 demonstrates the result of this simulation. It is worthwhile to mention that the upcoming results have been obtained assuming constant temperature boundary conditions on the bottom surface of the substrate. In other words it has been assumed that we have sufficient cooling power to maintain the bottom surface of the substrate at 300K.

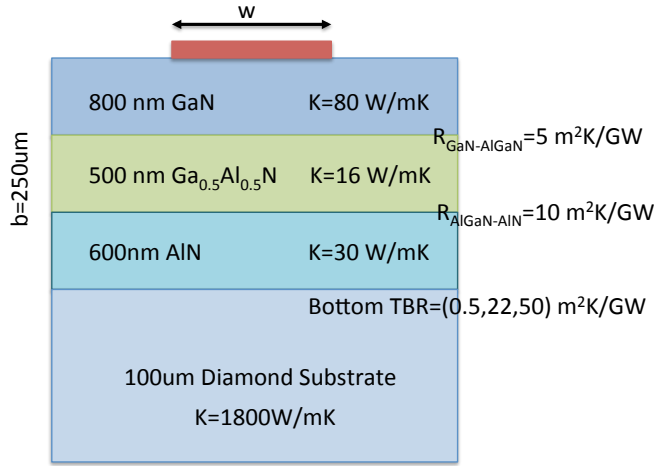


Figure 8: A HEMT multilayer structure on diamond substrate.

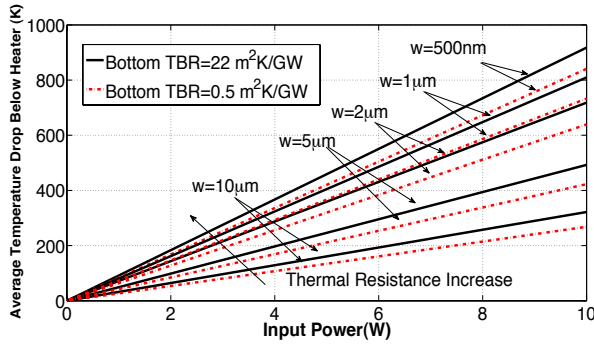


Figure 9: Temperature (K) drop vs. input power (W) for various gate lengths and two different bottom TBRs. Gate lengths (w) are substantially smaller compared to gate-to-gate spacing ($L \gg w$).

The above graph shows that the average temperature drop below the heater is a linear function of the power input, which suggests that we define the total stack resistance as the slope of the curves. The reason we prefer to report our results in terms of stack resistance instead of temperature is that the resistance is a more generalized parameter. Temperature rise only applies for a single input power. One can calculate the average temperature by multiplying the resistance by the desired power input.

Figure 9 compares stacks with different gate lengths and bottom thermal boundary resistances. As expected, devices with wider gates and lower thermal boundary resistances have lower thermal resistance. It should be noted that this plot is only for the two-dimensional case in which $L \gg w$.

We can investigate the effect of device width ($2L$) on thermal resistance by simply considering two limiting cases. First case represents the one-dimensional case ($L=w$) and the other, represents the semi-infinite two-dimensional case ($L \gg w$). As expected and shown in figure 10, devices with larger device width ($2L$) compared to their gate length (w) have less thermal resistance.

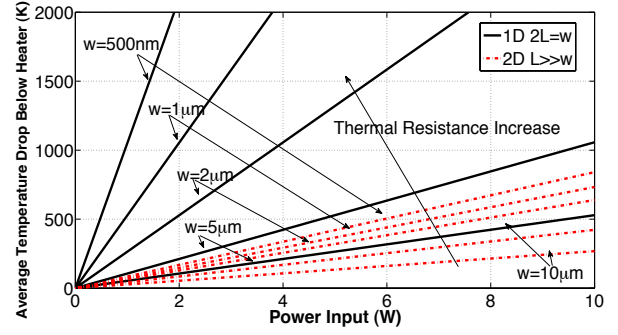


Figure 10: Comparison between 1D and 2D thermal resistance for a HEMT structure similar to figure 10 and bottom TBR of $0.5 \text{ m}^2\text{K/GW}$.

In the next step, we are interested in investigating the effect of device width on thermal resistance. In this investigation we take the device width ($2L$) to be sufficiently large compared to the gate length. In other words, we are focusing on the two dimensional semi-infinite case.

Now we compare the thermal resistance of two different multilayered structures for various lengths. One of these structures is shown in figure 8 with Diamond as its substrate. The other structure is shown in the figure below.

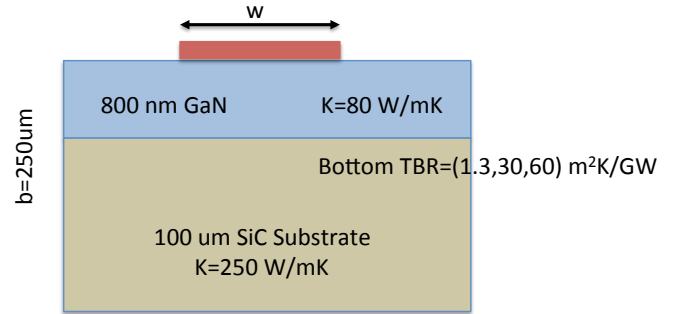


Figure 11: A HEMT structure on SiC substrate.

As shown above, the second structure has fewer layers and lies directly on a SiC substrate. Figure 12 compares these structures for various values of gate lengths and bottom TBRs. As expected, devices with larger gate lengths have lower thermal resistances. An interesting observation is that although diamond has much better thermal conductivity compared to SiC, they both have approximately equal stack resistance at low gate lengths and high bottom thermal boundary resistances. Even more interesting is that at low bottom TBRs the SiC stack has less thermal resistance compared to the diamond stack. This might be because the diamond substrate structure has two layers in addition to GaN. Not only do these two layers have less thermal conductivity than GaN, but also they introduce additional thermal boundary resistances into the system. Moreover, at lower gate lengths, each extra layer introduces an extra dispersion resistance that eventually causes the diamond substrate structure's thermal performance to decrease. However, at higher gate lengths, the dispersion

resistance becomes negligible and the diamond structure has lower stack resistance.

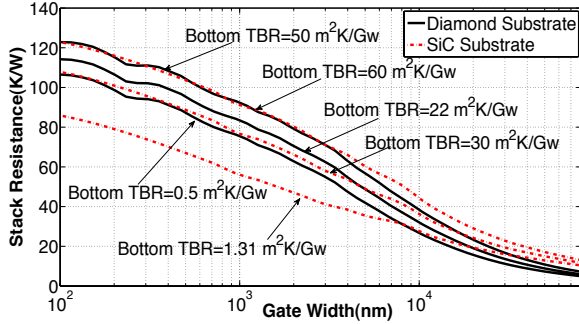


Figure 12: Resistance (K/W) vs. gate length (nm) for diamond (figure 8) and SiC (figure 11) substrate structures. Data points slightly smoothed with a Gaussian with a standard deviation of one data point and total width of three data points.

It should be noted that since the data points are logarithmically spaced, some noise exists at lower gate lengths. This is mainly because these points are very close in real space. Since there is some error inherent in our simulation, we see a slightly noisy behavior at these gate lengths. In order to see the trends more clearly, the plots have been smoothed via a Gaussian with a standard deviation of one data point and total width of three data points.

We now turn our attention to a HEMT structure with Si as its substrate (Fig. 13). Figure 14 demonstrates that the Si substrate structure has a trend similar to the SiC (Fig. 11) and diamond (Fig. 8) structures. At lower gate lengths, the Si structure has slightly greater stack resistance. We also see, for certain values of bottom TBR, both structures have the same stack resistance. However, at high gate lengths, the diamond structure performs better no matter the thermal boundary resistance.

Again, we can hypothesize that this behavior is mainly because geometrical dispersion resistance at lower gate lengths dominates the stack thermal resistance.

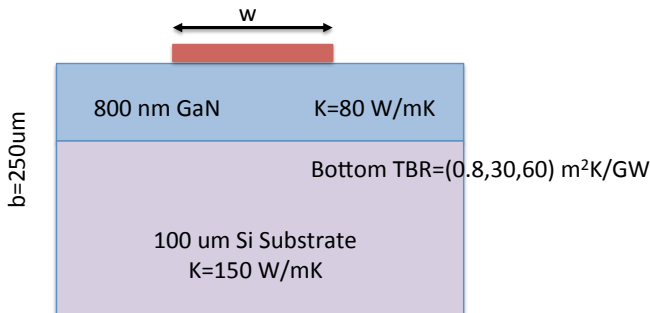


Figure 13: A HEMT structure on Si substrate.

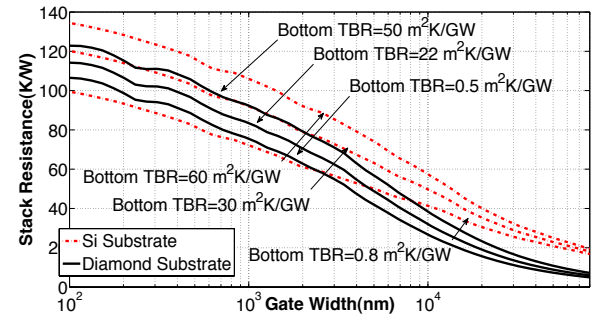


Figure 14: Resistance (K/W) vs. gate length (nm) for diamond (figure 8) and Si (figure 13) substrate structures. Data points slightly smoothed with a Gaussian with a standard deviation of one data point and total width of three data points.

Finally, we investigate the new generation HEMT device with GaN lying directly on diamond substrate (Fig. 15), and compare the expected stack resistance with those obtained for figures 11 and 13. The higher bound assumed for the thermal boundary resistance of GaN-diamond interface, correlates well with the values reported in table III of [12] for second-generation devices. Figure 16 shows the result of this comparison. As expected, in equal conditions, diamond substrates perform better in both high and low gate length regions.

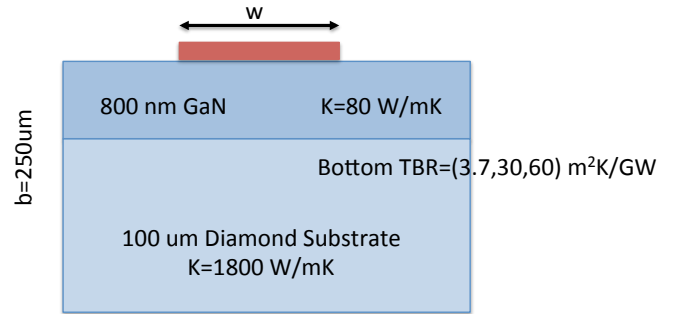


Figure 15: A new generation HEMT device with GaN directly on top of diamond.

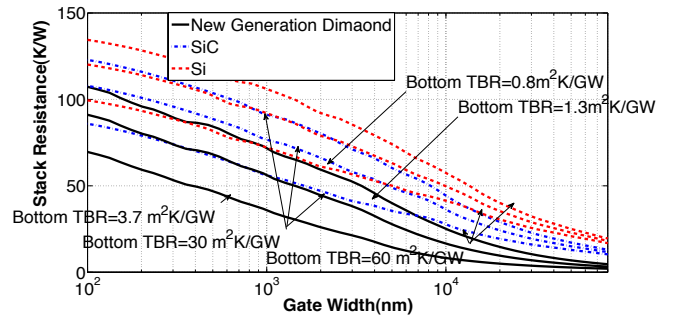


Figure 16: Resistance (K/W) vs. gate length (nm) for new generation diamond (figure 15), Si (figure 13) and SiC (figure 11) substrate structures. Data points slightly smoothed with a Gaussian with a standard deviation of one data point and total width of three data points.

b) Frequency-Domain Analysis

In this section we analyze the frequency response of a sample stack. To do this, we first apply a sinusoidal power to the gate and analyze the amplitude of the response (In this case the amplitude of the fluctuation of the maximum temperature). If we do this procedure for the stack shown in figure 15, (for two specific lengths), we obtain the following plot.

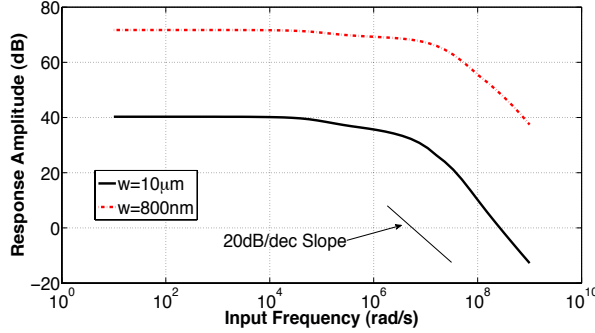


Figure 17: Amplitude Bode diagram for a stack of figure 18 with a $3.7 \text{ m}^2\text{K/GW}$ bottom TBR.

In figure 17, we observe that the slope of the high frequency curve is approximately -20 dB/dec , which is in fact the characteristic slope for systems governed by a PDE that has first partial derivative in time [13]. Moreover, the thermal properties of the thin GaN layer can be extracted from the high frequency portion of the plot whereas the low frequency portion of the plot represents the thermal properties of the thick diamond substrate.

Furthermore, we can extract the system's characteristic time from the breaking frequency of the above plot:

$$\tau \propto RC_{\text{store}} = \frac{1}{\omega_c} \quad (5)$$

where C_{store} represents the thermal storage capacity. It can be observed that the two stacks (with two different lengths) have approximately the same breaking frequency. Since the stack with the higher length has a higher capacity, it can be deduced that the resistance of the higher length stack is less than the resistance of the lower length stack, which conforms to the results of figure 16.

SUMMARY, DISCUSSION AND FUTURE WORK

In this paper, we validated our numerical model by comparing it to the one and two dimensional steady state and transient solutions. After that, we considered various HEMT structures on different substrates and with varying transition layer-substrate TBRs.

In comparing the thermal resistances of various stacks, we determined that at lower gate lengths, a geometrical dispersion resistance dominates the stack thermal resistance. This resistance diminishes as we approach higher lengths and the substrate resistance dominates the stack resistance at these lengths. Further, as figure 16 demonstrated, HEMT devices built directly on diamond outperform all other substrates at all

gate lengths. First generation diamond structures, similar to figure 8, performs better in high gate length scenarios, but perform comparably to Si and SiC substrates in lower gate lengths. Finally, we performed frequency analysis on the GaN-on-diamond HEMT structures for two different gate lengths. This analysis yielded the characteristic time of the stack. Furthermore, as in 3ω measurements, such a curve can be used to extract thermal properties of the stack. We intend to expand the capabilities of our model to include material anisotropies, temperature dependent thermal properties, and temperature dependent boundary resistances in the future.

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APPENDIX

This appendix details the validation of our model. We shall obtain the analytical solutions of certain one and two-dimensional steady and transient problems, and compare them with our simulation results.

a) One-Dimensional Steady State Validation:

We take the configuration below (figure A1) as the case study for our one-dimensional validation. It should be noted that in order to reach the 1D limiting case, we should have uniform heating along the whole width of the top surface, that is, the length of the gate (w) must be equal to the device width ($2L$).

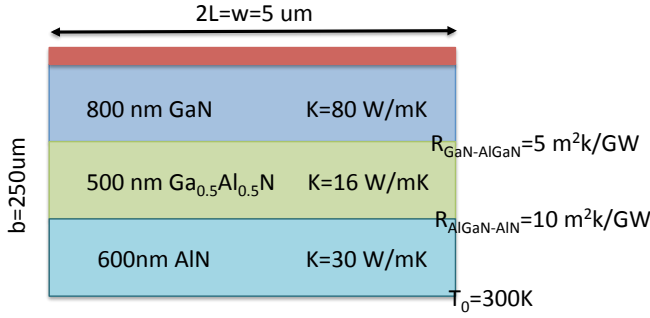


Figure A1: A typical multilayered HEMT structure (on type of diamond substrate). As our purpose in this point is only validation, the substrate has been replaced with a constant temperature boundary condition.

For the power input, we refer to the case study in reference 2. There; a total power input of (20V x 670 mA) was assumed for a device with 8 gates. Hence, we take on eighth of that value (1.675W) to be the power input for our single gate. The simulation result for this limiting case is shown below:

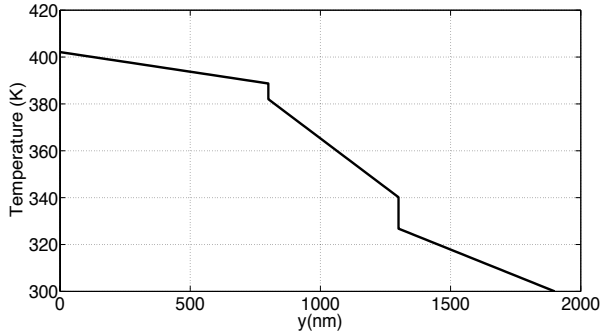


Figure A2: Temperature (K) vs. y (nm) plot for the stack shown in figure A1.

A comparison with the 1D resistance model (figure A3) shows that the 1D limiting case of the simulation agrees well with the resistance model:

$$\left. \begin{aligned} \Delta T_{\text{resistor}} &= R_{\text{tot}} q = 102.175 \text{ K} \\ \Delta T_{\text{simulation}} &= 102.091 \text{ K} \end{aligned} \right\} \Rightarrow \text{error} \approx 0 \quad (\text{A1})$$

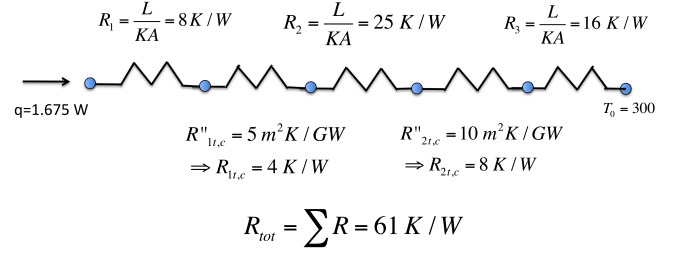


Figure A3: One dimensional resistance model for the stack shown in figure A1.

b) One Dimensional Transient Validation:

For this purpose we compare the simulation results to the solution of the following transient equation. As for the steady state one-dimensional case, the length of the gate (w) is equal to the device width ($2L$). For simplicity, we consider the geometry shown in figure A4.

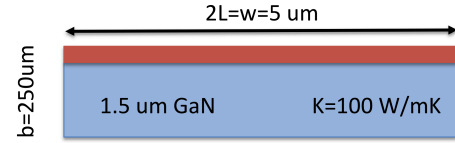


Figure A4: Geometry used for one-dimensional transient and two dimensional steady state and transient validation.

$$\begin{aligned} u_t &= \alpha u_{yy} \\ \begin{cases} u_y(y=0, t) = \frac{-q}{kA} \\ u(y=l, t) = 0 \\ u(y, t=0) = 0 \end{cases} \end{aligned} \quad (\text{A2})$$

Where

$$u = T - T_{\infty} \quad (\text{A3})$$

We realize that the above equation is a nonhomogeneous equation in space. One way to solve this problem is to obtain the steady state solution and subtract this solution from u :

$$\begin{aligned} 0 &= u_{yy,ss} \\ \begin{cases} u_y(0) = \frac{-q}{kA} \\ u(d) = 0 \end{cases} \end{aligned} \Rightarrow u_{ss} = \frac{q}{kA} (d - y) \quad (\text{A4})$$

Defining $\tilde{u}$ as:

$$\tilde{u} = u - u_{ss} \quad (\text{A5})$$

We have:

$$\begin{cases} \tilde{u}_t = \alpha \tilde{u}_{yy} \\ \tilde{u}_y(y=0, t) = 0 \\ \tilde{u}(y=d, t) = 0 \\ \tilde{u}(y, t=0) = \frac{q}{kA}(y-d) \end{cases} \quad (\text{A6})$$

Using separation of variables we obtain:

$$\tilde{u} = \sum_{n=0}^{\infty} -\frac{8dq}{kA(2n+1)^2\pi^2} \cos\left(\frac{(2n+1)\frac{\pi}{2}}{d}y\right) \exp\left(-\frac{(2n+1)^2\frac{\pi^2}{4}\alpha}{d^2}t\right) \quad (\text{A7})$$

Thus:

$$u = \sum_{n=0}^{\infty} -\frac{8dq}{kA(2n+1)^2\pi^2} \cos\left(\frac{(2n+1)\frac{\pi}{2}}{d}y\right) \exp\left(-\frac{(2n+1)^2\frac{\pi^2}{4}\alpha}{d^2}t\right) + \frac{q}{kA}(d-y) \quad (\text{A8})$$

As can be seen in the figure below, the analytical and simulation results agree to a very good extent such that they are practically indistinguishable.

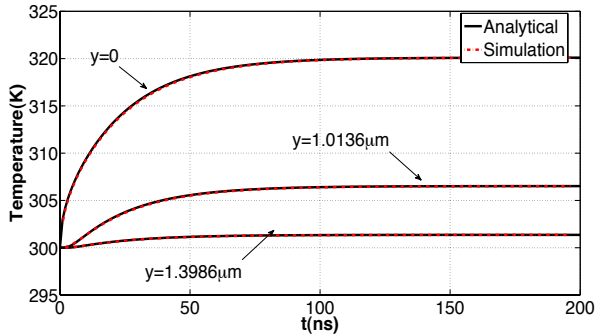


Figure A5: Temperature (K) vs. t (ns) for various depths. This plot illustrates the analytical solution and the simulation results of a 1D transient problem.

c) Two-Dimensional Steady State Validation:

As done in the previous cases, we compare the analytical solution of a sample heat equation to our simulation results. If we consider the following equation for the geometry of figure A4:

$$\begin{cases} 0 = u_{yy} + u_{xx} \\ u_x(x=0, y) = u_x(x=w, y) = 0 \\ u_y(x, y=0) = \frac{-q}{kA} \\ u(x, y=d) = 0 \end{cases} \quad (\text{A9})$$

It can easily be shown that the solution to the above equation is linear and this 2D problem actually resembles the linear case considered in part (a). Hence, we consider the following non-trivial boundary conditions (triangular heat flux distribution along the top surface):

$$\begin{cases} 0 = u_{yy} + u_{xx} \\ u_x(x=0, y) = u_x(x=w, y) = 0 \\ u_y(x, y=0) = \frac{-q}{kAw}x \\ u(x, y=d) = 0 \end{cases} \quad (\text{A10})$$

It can be shown that the above equation has the following solution:

$$u = \sum_{n=1}^{\infty} \frac{2qw(1-(-1)^n)}{n^3\pi^3Ak} \left[\sinh\left(\frac{n\pi}{w}y\right) - \tanh\left(\frac{n\pi}{w}d\right) \cosh\left(\frac{n\pi}{w}y\right) \right] \cos\left(\frac{n\pi}{w}x\right) + \frac{q}{2kA}(d-y) \quad (\text{A11})$$

The following figures clearly show that the simulation results agree well with the analytical solution.

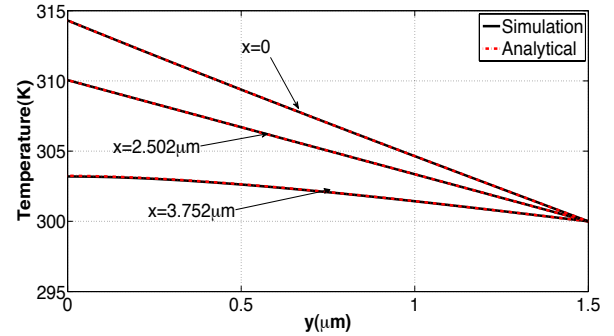


Figure A6a: Temperature (K) vs. y (μm) for various constant x values. This plot illustrates the analytical solution and the simulation results of a 2D steady state problem with triangular input heat distribution along the top surface.

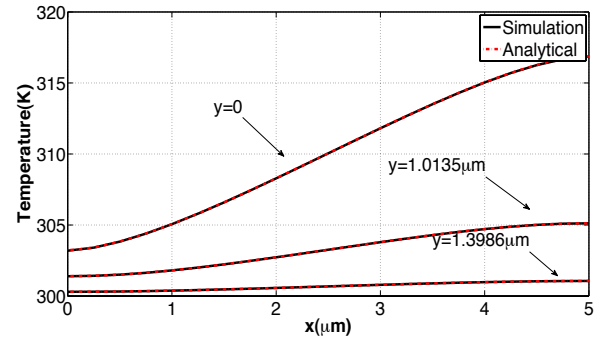


Figure A6b: Temperature (K) vs. x (μm) for various constant y values. This plot illustrates the analytical solution and the simulation results of a 2D steady state problem with triangular input heat distribution along the top surface.

d) Two-Dimensional Transient Validation:

We now compare our simulation results to the solution of the following transient two-dimensional heat equation for the geometry of figure A4. Again we consider triangular input heat distribution along the top surface in order to avoid the trivial one dimensional solution.

$$u_t = \alpha(u_{yy} + u_{xx})$$

$$\begin{cases} u_x(x=0, y, t) = u_x(x=w, y, t) = 0 \\ u_y(x, y=0, t) = \frac{-q}{kAw}x \\ u(x, y=d, t) = 0 \\ u(x, y, t=0) = 0 \end{cases} \quad (A12)$$

The above equation has the following solution:

$$u = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} D_{mn} \cos\left(\frac{(2m+1)\pi}{2d}y\right) \cos\left(\frac{n\pi}{w}x\right) \exp(-\lambda^2 t) + \sum_{m=0}^{\infty} D_{m0} \cos\left(\frac{(2m+1)\pi}{2d}y\right) \exp(-\lambda^2 t) + u_{ss} \quad (A13)$$

Where

$$D_{mn} = -\frac{32dqw^2 \sin^2\left(\frac{n\pi}{2}\right)}{\pi^4 An^2 k(4d^2 n^2 + (2mw + w)^2)} \quad (n \neq 0) \quad (A14)$$

$$D_{m0} = \frac{4dq}{Ak(2\pi m + \pi)^2}$$

$$\lambda^2 = \alpha(\beta^2 + \theta^2) = \alpha\left(\frac{n^2 \pi^2}{w^2} + \frac{(2m+1)^2 \pi^2}{4d^2}\right) \quad (A15)$$

$$u_{ss} = u(eqA11) \quad (A16)$$

Again, as the plot shows, there is excellent agreement between the analytical solution and the numerical simulation.

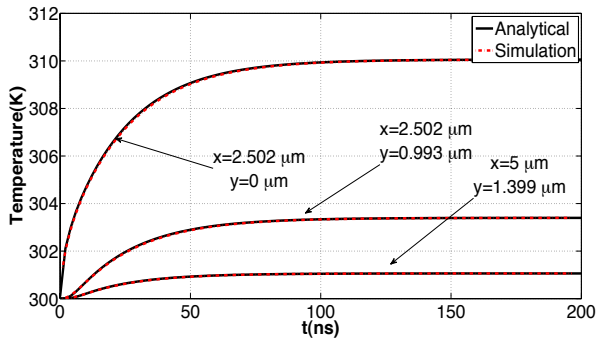


Figure A7: Temperature (K) vs. t (ns) for various points. This plot illustrates the analytical solution and the simulation results of a 2D transient problem with triangular input heat distribution along the top surface.